Variational Adaptive Correlation Method for Flow Estimation

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Abstract—A variational approach is presented to the estimation of turbulent fluid flow from particle image sequences in experimental fluid mechanics. The approach comprises two coupled optimisations for adapting size and shape of a Gaussian correlation window at each location and for estimating the flow, respectively. The method copes with a wide range of particle densities and image noise levels without any data-specific parameter tuning. Based on a careful implementation of a multiscale nonlinear optimisation technique we demonstrate robustness of the solution over typical experimental scenarios and highest estimation accuracy for an international benchmark dataset (PIV challenge).

EDICS Category: ARS-IVA

I. INTRODUCTION

A. Overview

Particle image velocimetry (PIV) is a non-intrusive optical measurement technique for industrial fluid flow questions [1], [2]. Small particles are added to liquids and gases and act as an indicator for the movement of the investigated substance around obstacles or in regions where flows mix. For the two-dimensional variant of PIV, a thin sheet of the volume is illuminated by a laser light, rendering the particles therein visible. A high speed camera system records an image sequence of the highlighted area. The experimental setup is sketched in Fig. 1.

The analysis of two consecutively recorded frames allows to determine the movement of particles, and in this way to measure velocity, turbulence or other derived physical properties of the fluid. Figure 2 presents an example for the image data and the derived vector field.

Cross-correlation has developed as the state of the art method for analysing PIV image data due to its robustness against distortions typical for this application [3]. These are high image noise levels caused by high shutter speeds, as well as varying laser output and particles leaving or entering the illuminated plane.

In this work we formulate a variational approach to velocity measurement based on cross-correlation. Furthermore, we replace the usually employed sharp square correlation window by a Gaussian weighting function. A mathematically sound and novel criterion to adapt window sizes and shapes is proposed, which directly formulates the aim to improve the accuracy of the velocity estimation, especially in the presence of vortices as demonstrated in Fig. 3.

Both displacement estimation and window adaptation are joint into a coupled optimisation problem and solved using methods for non-linear and non-convex optimisation.

We demonstrate the ability of the approach to significantly improve the accuracy of flow measurements with synthetic examples as well as real turbulent data. For an international benchmark data set our method outperforms most of the concurring approaches.

B. Related Work and Contribution

A vast body of literature exists on all aspects of the application of cross-correlation for analysing PIV image data, here we only refer to [1] for an excellent overview. In a typical implementation, an exhaustive search over integer displacements is performed to find the highest correlation peak which corresponds to the most probable displacement of the considered region between two consecutive frames. This process can be speeded up using Fast Fourier Transformation. The correlation function is then interpolated to gain sub-pixel accuracy. In contrast, we present a variational approach which determines the displacements by continuously maximising the cross-correlation measurement.
optical flow based methods and cross-correlation: an optical flow approach with physically sound regularisation terms, which penalise large variations in the rotation and divergence of the flow, is endowed with an additional data term. Similarity to a coarse vector field originating from a local correlation approach is enforced. The displacement field estimated using cross-correlation is used to initialise a variational optical flow approach in [11]. For a comprehensive synopsis on variational methods for fluid flow measurement we refer to [12].

Much effort has been put into improving the spatial resolution of cross-correlation methods [13], [14] by replacing the fixed square interrogation windows by appropriate alternatives. The authors of [15] investigate a class of cone-shaped weighting functions and optimise the shape parameters by means of the frequency-response, however not with respect to a specific image data set. In [16], [17] the size of square windows is locally adapted to the signal quality and spatial fluctuations in the flow. Window adaptation is used in [18] at interfaces to fixed objects in the scene. The authors of [19] propose a criterion based on the flow gradients and image quality to select the optimal shape of an elliptical window. In [20] a Gaussian weighting function is stretched and rotated along the measured mean displacement. Gaussian weights are used both in a local [21] and global context [22] for smoothing the optical flow constraint, however with isotropic windows of fixed size common for all positions. In our work, the correlation window is also described by a “soft” Gaussian weighting function. However, we formulate a sound criterion for the location-dependent choice of the window shape parameters by means of an error model function. The window adaptation consists of finding the window shape which minimises the predicted measurement error.

Our contribution is a variational formulation for a correlation-based approach for measuring motion in PIV image pairs. A Gaussian weighting function controls the image region considered in the displacement estimation. The shape of the window is controlled by means of a function which approximates the expected measurement error. Minimisation leads to the optimal window shape with respect to this error model. Displacement measurement and window adaptation are formulated as a pair of interdependent optimisation problems. It is solved via a multiscale gradient-based algorithm.

This work summarises and extends the results in [23]. An abridged version was published in [24] with the focus on image processing. In [25] we investigate our approach from the applied fluid mechanics point of view.

C. Organisation

In Sect. II we formulate our approach to adaptive fluid flow measurement as a continuous optimisation problem. Section III details on the discretisation and the employed optimisation method. We verify both in the experimental section (Sect. IV) and conclude in Sect. V.

II. APPROACH

A. Problem Definition

Given a pair of images, \( g_1, g_2 \), defined on the image domain \( \Omega \subset \mathbb{R}^2 \), we are interested in a vector field \( \mathbf{u} : \Omega \rightarrow \mathbb{R}^2 \).
which locally describes the displacement of image structures from $g_1$ to $g_2$.

### B. Variational Correlation

For the estimation of the displacement at $x \in \Omega$, we consider the following optimisation problem:

$$
\min_{u \in \mathbb{R}^2} C(u, \Sigma, x), \quad \text{with} \quad C(u, \Sigma, x) = -\int \mathcal{W}(y-x, \Sigma) g_1 \left(y - \frac{u}{2}\right) g_2 \left(y + \frac{u}{2}\right) \, dy
$$

Basicallly, the considered regions in $g_1$ and $g_2$ are shifted until they fit best with respect to the correlation measurement. The function $\mathcal{W}(x, \Sigma) \in [0,1]$ weights the impact of the image region in the vicinity of $x$. In Sect. II-C we provide further details and propose a method to adapt the parameter $\Sigma$ which controls shape and size with the aim to improve accuracy.

The objective function $C(u, \Sigma, x)$ is non-linear and highly non-convex in $u$ as Fig. 4 illustrates. Thus, the employed optimisation method has to circumnavigate several local optima to reach the correlation peak. We will come back to this issue in Sect. III-B.

Finally, we extend the problem definition to estimating a complete flow field on $\Omega$ and define the variable domains

$$
\mathcal{U} := \left\{ u : \Omega \mapsto \mathbb{R}^2 \left| \int_{\Omega} \| u(x) \|_2^2 \, dx < \infty \right. \right\}
$$

and

$$
\mathcal{S} := \left\{ \Sigma : \Omega \mapsto S \left| \int_{\Omega} \| \Sigma(x) \|_2^2 \, dx < \infty \right. \right\}.
$$

The set $S \subset S^2_{++}$ denotes the allowed window shapes as detailed further below, where $S^2_{++}$ is the set of symmetric, positive $2 \times 2$ matrices. $\| \cdot \|_F$ denotes the Frobenius matrix norm. For given, fixed window shapes $\Sigma$, we determine a flow field $u$ on $\Omega$ by solving the following optimisation problem:

$$
\min_{u \in \mathcal{U}} C(u, \Sigma)
$$

with $C(u, \Sigma) := \int_{\Omega} C(u(x), \Sigma(x), x) \, dx$

### C. Window Adaptation

Instead of a square or rectangular window, we choose the weighting function as a non-normalised Gaussian function:

$$
\mathcal{W}(x, \Sigma) := \frac{G(\Sigma, x)}{G(\Sigma, 0)} = \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right), \quad (7)
$$

with $G(\Sigma, x) = \frac{1}{2 \pi \sqrt{\det \Sigma}} \exp \left(-\frac{1}{2} x^T \Sigma^{-1} x\right)$ being the normalised two-dimensional Gaussian function with covariance matrix $\Sigma \in S^2_{++},$ centred at $x = 0$. The shape parameter $\Sigma$ allows to select the image region considered for the correlation measurement. It provides high flexibility in three degrees of freedom, which can be interpreted as geometric properties of the ellipsoidal contour line $\{x \in \mathbb{R}^2 \mid \mathcal{W}(x, \Sigma) = \exp(-1)\}$, namely semi-minor radius $r > 0$, anisotropy $a \in [0,1]$ and orientation $\alpha \in [0, \pi)$.

$$
\Sigma(r, a, \alpha) = \frac{1}{2} QDQ^T
$$

with $Q = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$ and $D = \text{diag} \left( \frac{r^2}{1 - a^2}, r^2 \right)$.

Figure 5 illustrates the influence of the three parameters. However, in the following we parametrise the window shape by the matrix representation $\Sigma$.

The choice of the window shape is motivated by the following consideration: when cross-correlation is employed for estimating displacements, it is implicitly assumed that motion within the considered image regions is uniform. This, however, only holds true in very simple cases and leads to large estimation errors in regions with motion gradients as they occur in typical fluid flows. Decreasing the window size reduces this effect, however at the costs of a smaller supporting area, a smaller number of observed particles and thus a higher impact of image noise.

With the improvement of measuring accuracy in mind, we model the described effects and find a trade-off between them by solving the following optimisation problem for each location $x \in \Omega$

$$
\Sigma(x) \in \arg \min_{\Sigma \in \mathcal{S}} E(\Sigma, u, x)
$$

with $E(\Sigma, u, x) := E_{\text{homog}}(\Sigma, u, x) + E_{\text{noise}}(\Sigma), \quad (9)$

assuming that a displacement field $u$ is given. Furthermore, we incorporate lower ($\lambda_{\text{min}}$) and upper ($\lambda_{\text{max}}$) bounds
on the window size by defining the constraint set \( S := \{ \Sigma \in S^2_{++} \mid \lambda_{\min} I \preceq \Sigma \preceq \lambda_{\max} I \} \). \( I \) denotes the identity matrix of appropriate size and \( S \preceq T \) indicates positive semi-definiteness of \( T - S \) for symmetric matrices \( S, T \).

The error model function \( E(\Sigma, u, x) \) is composed of the following two terms:

\( a) \) Homogeneity Term: The first part of the objective (9) describes the error caused by the violation of the assumption that the observed motion within the chosen window is homogeneous:

\[
E_{\text{homog}}(\Sigma, u, x) := \int_{\mathbb{R}^2} w(y - x, \Sigma) e(x, y, u) \, dy \tag{10}
\]

\[
e(x, y, u) := \left\{ \begin{array}{ll}
\|u(y) - u(x)\|_2^2 & \text{if } y \in \Omega \\
\epsilon_{\text{outside}}^2 & \text{otherwise}
\end{array} \right.
\tag{11}
\]

The function \( e(x, y, u) \) measures the squared Euclidean distance of \( u(y) \) to the displacement in the point of interest, \( u(x) \), while a constant error value is assumed for regions outside the image domain \( \Omega \). The errors are weighted by the window function which is parameterised by \( \Sigma \).

\( b) \) Noise Term: The second term in (9) describes the impact of image sensor noise and unpaired particles on the accuracy. We define it as

\[
E_{\text{noise}}(\Sigma) = \frac{\sigma^2}{2\pi \sqrt{\det \Sigma}}, \tag{12}
\]

where \( \sigma \) is a parameter which describes the image noise level. Intuitively spoken, this term describes the expectation that the error reduces when the measurement support, \( \int w(x, \Sigma) \, dx = 2\pi \sqrt{\det \Sigma} \) increases for larger windows \( \Sigma \). A more detailed derivation of this term can be found in the Appendix.

\( c) \) Global Window Adaptation: Finally, we extend the local window estimation to

\[
\min_{\Sigma \in S} E(\Sigma, u) \quad \text{with } E(\Sigma, u) := \int_{\Omega} E(\Sigma(x), u, x) \, dx,
\]

which optimises the window shapes globally in terms of the matrix-valued function \( \Sigma \in \mathcal{S} \).

\section{D. Joint Approach}

In Sect. II-B we introduced a motion estimation approach and presumed that the window parameters are given. In contrast, in Sect. II-C we fixed a displacement field and adapted the correlation windows to it. We describe this chicken-and-egg-dependency as a mathematically tractable problem:

\[
u^* \in \arg \min_{u \in \mathcal{U}} C(u, \Sigma^*) \quad \text{and} \quad \Sigma^* \in \arg \min_{\Sigma \in \mathcal{S}} E(\Sigma, u^*) \tag{13}
\]

The two optimisation problems have non-linear and non-convex objective functions each, are interconnected through the variables \( \Sigma^* \) and \( u^* \), and thus have to be solved jointly.

Note, that \( C(u, \Sigma^*) \) is only minimised with respect to \( u \), but not \( \Sigma^* \), as the window shapes should only be steered by the error model function and not by the correlation measurement. If image data should be considered in the window choice, an additional term should be incorporated into \( E(\Sigma, u^*) \).

\section{III. DISCRETISATION AND OPTIMISATION}

A number of carefully chosen approximations and relaxations were applied to make the optimisation problem (13) tractable.

\subsection{A. Discretisation}

The functions \( u \) and \( \Sigma \) are discretised component-wise on a regular grid \( X_V \) with spacing \( a_V \) at coordinates \( x_i \in X_V \). Furthermore, we define \( u_i := u(x_i) \) and \( \Sigma_i := \Sigma(x_i) \). Using finite elements with piecewise linear basis functions \( \phi_i(x) \), we approximate the functions as

\[
u(x) \approx \sum_{x_i \in X_V} \phi_i(x) u_i \quad \text{and} \quad \Sigma(x) \approx \sum_{x_i \in X_V} \phi_i(x) \Sigma_i.
\]

Note that it is possible to extend the method to arbitrary grids, e.g. irregular ones that adapt to the seeding density, as it is used in [26]. The integrals in \( C \) and \( E \) are also discretised using the introduced finite elements:

\[
C(u, \Sigma) \approx \sum_{x_i \in X_V} \int_{\Omega} \phi_i(x) \, d \Omega \quad C(u_i, \Sigma_i, x_i)
\]

\[
E(\Sigma, u) \approx \sum_{x_i \in X_V} \int_{\Omega} \phi_i(x) \, d \Omega \quad E(\Sigma_i, u_i, x_i)
\]

Note that \( A_i := \int_{\Omega} \phi_i(x) \, d \Omega \) evaluates to \( a_V^2 \), almost everywhere. The nested integrals in \( C(u, \Sigma, x) \) and \( E_{\text{homog}}(\Sigma, u, x) \) are discretised accordingly.

For windows of reasonable size the function \( w(x, \Sigma) \) incorporated in both terms weights only few terms with considerable impact. In order to reduce computational effort, we limit evaluation to a bounding box which contains all \( y \in \Omega \), such that \( w(y - x, \Sigma) \leq 10^{-3} \).

The image data \( g_1, g_2 \) is given on a regular grid with a spacing typically smaller than \( a_V \). They are transferred into a cubic spline representation with all values outside the image domain defined to be zero. Using an efficient implementation based on [27], it is possible to evaluate the function value \( g_i \), its gradient \( \nabla g_i \) and second derivatives \( H g_i \).

\subsection{B. Optimisation}

\( 1) \) Barrier Function: The constraints \( \Sigma \in S \) are incorporated into the energy function using logarithmic barriers:

\[
B_S(\Sigma) := -\mu (\log \det(\Sigma - \lambda_{\min} I) + \log \det(\lambda_{\max} I - \Sigma)).
\]

The penalty weight is \( \mu := 10^{-2} \) throughout the work. Then we minimise \( E_S(\Sigma, u, x) := E(\Sigma, u, x) + B_S(\Sigma) \) instead of (9), which is up to the symmetry of \( \Sigma \) – an unconstrained problem.

\( 2) \) Single Scale Optimisation: A major simplification of the problem is to replace both minimality objectives by the stationary conditions

\[
\nabla_u C(u, \Sigma) = 0 \quad \forall x_i \in X_V \tag{14a}
\]

\[
\text{and} \quad \nabla_\Sigma E(\Sigma, u) = 0 \quad \forall x_i \in X_V. \tag{14b}
\]

A Newton step with respect to all displacement and window shape variables is employed to find a set of \( \Sigma \in \mathcal{S} \) and \( u \in \mathcal{U} \).
that satisfy these conditions. It is extended by a line search method
to avoid local maxima and saddle points.

Note, that although each equality constraint in (14a)-(14b) is
a nonlinear and non-convex function in both \( u \) and \( \Sigma \), they
strongly simplify to
\[
\nabla_v C(u, \Sigma) = \nabla_u A, C(u_i, \Sigma_i, x_i) = 0 \quad \forall x_i \in X_v \quad \nabla_v E(\Sigma, u) = \nabla u A, E(\Sigma_i, u, x_i) = 0 \quad \forall x_i \in X_v .
\]

Thus, the displacements can be updated independently of each
other which is a consequence of the fact that we did not add
a spatial regularisation term on \( u \). In the same way, the
refinement of the window shape parameters is independent in
the coordinates. The optimisation loop can be summarised as:

\[
\begin{align*}
\text{procedure SINGLESCALE} & \text{SOLUTION}(u^{(1)}, \Sigma^{(1)}) \\
\quad & \begin{array}{l}
k \leftarrow 1 \\
\quad \text{repeat} \\
\quad \quad \text{for all } x_i \in X_v \text{ do} \\
\quad \quad \quad u^{(k+1)}_{i} \leftarrow \text{VARIABLE} \text{UPDATE}(C(u, \Sigma), u_i, (u^{(k)}, \Sigma^{(k)})) \\
\quad \quad \Sigma^{(k+1)}_{i} \leftarrow \text{VARIABLE} \text{UPDATE}(E(\Sigma, u), u_i, (\Sigma^{(k)}, u^{(k)})) \\
\quad \quad \text{end for} \\
\quad k \leftarrow k + 1 \\
\quad \text{until stopping criterion fulfilled} \\
\quad \text{return } (\Sigma^{(k)}, u^{(k)})
\end{array}
\end{align*}
\]

An upper bound on the change of the variables is used as
stopping criterion. The function \text{VARIABLE} \text{UPDATE}
improves the solution \( x_0 \) with respect to \( f \) by updating a subset of
variables \( y \).

\[
\begin{align*}
\text{procedure VARI} \text{ABLE} \text{UPDATE}(f, g, x_0) \\
\quad & \begin{array}{l}
y \leftarrow \nabla_y f(x_0), H \leftarrow H_y f(x_0) \quad \triangleright \text{gradient, Hessian} \\
\Delta y \leftarrow -(H + \lambda I)^{-1} g \quad \triangleright \text{Newton step direction w.r.t. } y \\
\alpha \leftarrow \alpha_{\text{max}}, x_\alpha \leftarrow x_0 \\
\quad \text{while } \alpha \geq \alpha_{\text{min}} \text{ do} \\
\quad \quad x_\alpha|_y \leftarrow x_0|_y + \alpha \Delta y \quad \triangleright \text{update only variables } y \\
\quad \quad \text{if } f(x_\alpha) < f(x_0) \text{ then} \\
\quad \quad \quad \text{return } x_\alpha \\
\quad \quad \quad \alpha \leftarrow \beta \alpha \quad \triangleright \text{update failed}
\quad \quad \text{end while} \\
\quad \text{return } x_0 \\
\end{array}
\end{align*}
\]

The parameters were chosen conservatively: \( \alpha_{\text{min}} = 10^{-9}, \alpha_{\text{max}} = 1 - 10^{-3}, \beta = 10^{-1} \) and \( \lambda = 100 \).

3) Multiscale Optimisation: As indicated by Fig. 4, the
problem has many local minima which we intend to
circumnavigate by wrapping a multiscale framework around the
optimisation loop. To this end, we represent the problem at the
original as well as a couple of coarser resolutions. The grid
spacings (data and variable) enlarge by factor \( s > 1 \) when
descending one level. E.g. for a 5-level dyadic pyramid, we
have \( s = 2 \) and we denote the resolution scales – from finest
to coarsest – as \( \{1, 2, 4, 8, 16\} \).

The multiscale framework first recursively transfers image
and initial variable values from the finest to the coarsest level.
Then at each resolution the estimated solution of the next
coarser level act as initialisation for the variable refinement
in \text{SINGLESCALE} \text{SOLUTION}.

Displacement variables are re-sampled to finer or coarser
grids using cubic spline interpolation. A small binomial low-
pass filter is used to avoid aliasing while down-sampling.
The multiscale image representation is created with the same
technique. The re-sampling process of the window shape
parameter is slightly more complex, as the constraint \( \Sigma \in S \)
has to be conserved. However, simple component-wise bi-
linear interpolation guarantees that the re-sampled value lies in
the convex hull of the interpolated values, and thus in \( S \). The
same argument holds for applying low-pass filters as long as
their coefficients add up to one, such as it is the case for the
employed binomial filters before down-sampling. Given the
image data and (a possibly zero) initial solution, the overall
optimisation can be summarised as:

\[
\begin{align*}
\text{procedure MULTISCALE} & \text{SOLUTION}(g_{1}^{[1]}, g_{2}^{[1]}, u^{[1]}, \Sigma^{[1]}) \\
\quad & \begin{array}{l}
\quad \text{for } l = 2, 3, \ldots, l_{\text{max}} \text{ do} \quad \triangleright \text{fine to coarse} \\
\quad \quad \text{create } g_{l}^{[l]} \text{ by downsampling } g_{l-1}^{[l-1]} \text{. for } i \in \{1, 2\} \\
\quad \quad \text{create } u^{[l]} \text{ and } \Sigma^{[l]} \text{ by down-sampling } u^{[l-1]} \text{ and } \Sigma^{[l-1]} \\
\quad \text{end for} \\
\quad \text{for } l = l_{\text{max}}, l_{\text{max}} - 1, \ldots, 2 \text{ do} \quad \triangleright \text{coarse to fine} \\
\quad \quad (u^{[l]}, \Sigma^{[l]}) \leftarrow \text{SINGLESCALE} \text{SOLUTION}(u^{[l]}, \Sigma^{[l]}) \\
\quad \quad \text{create } u^{[l-1]} \text{ and } \Sigma^{[l-1]} \text{ by up-sampling } u^{[l]} \text{ and } \Sigma^{[l]} \\
\quad \text{end for} \\
\quad (u^{[1]}, \Sigma^{[1]}) \leftarrow \text{SINGLESCALE} \text{SOLUTION}(u^{[1]}, \Sigma^{[1]}) \\
\quad \text{return } (u^{[1]}, \Sigma^{[1]})
\end{array}
\end{align*}
\]

Further details on the implementation can be found in [23].

IV. EXPERIMENTS

In our experiments we investigated the basic properties of the
window adaptation (Sect. IV-A, IV-B), and evaluated the joint
approach with synthetic benchmark data (Sect. IV-C) as
well as real-world data (Sect. IV-D).

The proposed methods was implemented mostly in MAT-
LAB. Geometric properties of a window \( \Sigma \), such as radius, refer
to the level contour \( \{x \in \mathbb{R}^2 | u(x, \Sigma) = \exp(-1)\} \),
which is also used for visualisation. No additional displacement filters
(e.g. vector median) were applied.

A. Window Adaptation Strategies

The following experiments were designed to estimate the
total potential of the proposed error model to improve the accuracy
of the variational correlation method. As we want to concen-
trate on the suitability of error model function and not on errors
caused by the continuous optimisation process, we simplify the
method as follows: in order to avoid sub-optimal local minima,
the optimisation of the correlation is initialised by the ground
truth displacement. For the same reason, we do not adapt the
window continuously but evaluate the deviation from ground
truth for 975 window shapes using (8) with varying radius \( r \),
orientation \( \alpha \) and anisotropy \( \alpha \).

\[
S := \left\{ \Sigma(r, \alpha, \alpha) \left| \begin{array}{l}
r \in \left\{ 2^\frac{i}{4} \right\}_{i \in \{-4, -3, \ldots, 10\}} \\
\alpha \in \left\{ 1 - 2^\frac{i}{4} \right\}_{i \in \{0, 1, \ldots, 8\}} \\
\alpha \in \left\{ \frac{1}{\pi} \right\}_{i \in \{0, 1, \ldots, 7\}} \\
\end{array} \right. \right\} .
\]
B. Synthetic Vector Fields

For arbitrary vector fields, the optimal window shape with respect to the proposed error model can form complex structures. To demonstrate the behaviour of the window adaptation methods, we investigate a couple of simple synthetic vector fields. Only windows were updated, while displacements were kept fixed after initialisation. If not mentioned otherwise, in all experiments the initial window radius was 5, the upper radius limit was \( r_{\text{max}} = 0 \) pixels (px), and no lower limit was imposed. Furthermore, we chose \( \sigma = 1 \) and \( c_{\text{outside}} = 0 \).

Figure 8 illustrates the adapted windows in the presence of an affine flow, i.e. when \( u(x) \) can be written as an affine function in \( x \). If there are no further influences (such as the boundary terms in (b)), the resulting shapes only depend on the Jacobian of \( u \), more precisely on its outer product.

In transition zones, e.g. where flows of different direction meet, fixed windows are disadvantageous, because they smooth over high velocity gradients and thus wipe out details. Thus, we investigate this situation in four simplified scenarios. It becomes clear, that the window adaptation is invariant against a constant offset (Fig. 9a vs. 9b) and rotation (Fig. 9a vs. 9c) of the vector field, but only depends on the orientation of the gradient. The sinusoid-shaped vector field in Fig. 9c is motivated by the data set introduced in Fig. 6.

Finally, we investigate scenarios with sharp motion boundaries where it is of great importance for the measurement accuracy that the window adaptation process respects the flow discontinuities. The experiments in Fig. 10 combine round and square shaped boundaries with constant and affine vector fields. In any case, the adapted window shapes respect the boundaries well.

C. Synthetic PIV Benchmark Data Set

The data set shown in Fig. 6 and already investigated in Sect. IV-A was created for the PIV Challenge 2005 and used to evaluate the spatial resolution of 19 PIV algorithms. First experiments showed that it is essential to have a good initialisation for the proposed adaptive approach. To this
end we first estimated a rough displacement field using fixed windows and then process the data with window adaptation.

For the case *Sinusoids* I the initial vector field was calculated with 5 multiscale levels, using a scaling factor of $\sqrt{2}$ and round windows with radius 6. The adaptive approach used only 3 multiscale levels. We set $\sigma = \epsilon_{\text{outside}} = 20$, and constrained the windows radii to the range 2 to 40 px.

The results in Fig. 6 demonstrate that fixed windows can recover the overall structure but smooth over small details. In view of the following processing step, we favour a rough reconstruction over a more detailed but definitely noisy one which can be achieved using smaller windows. With the window adaptation enabled, even the structures at the smallest scale can be reconstructed well up to few outliers, as windows align perpendicular to the velocity gradients and along regions of homogeneous motion. Additional disturbances can

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**Fig. 7.** Comparison of window selection strategies: mean error of the fixed radius (dotted line), oracle (dashed line) and the error model based strategy (solid line). (a) Results for *Sinusoids* I, comparing the fixed radius $\mu_3(r)$ strategy for varying radii to the best values of the alternative strategies. (b) Same as (a), but for the error model strategy and varying parameter $\sigma$. Best results are $\mu_1^* = 0.00627$, $\mu_2^* = 0.0421$ (for $\sigma^* = 10$) and $\mu_3^* = 0.0796$ ($r^* = 2$). (c)-(d): Same as (a)-(b), but for the data set *Sinusoids* II, with $\mu_1^* = 0.0255$, $\mu_2^* = 0.109$ ($\sigma^* = 10^{7/4}$), $\mu_3^* = 0.206$ ($r^* = 25/2$).

**Fig. 8.** Synthetic displacement fields (arrows) and some of the adapted windows (ellipses). **Constant vector fields:** (a) The window radii would approach infinity due to the lack of a gradient, but is limited by the constraint $r \leq 6$. (b) The windows are additionally constrained to adapt to the boundaries of the image domain by setting $\epsilon_{\text{outside}} = 10$. **Affine vector fields:** (c) Rotational field with isotropic gradients leads to round windows. (d) Vector field with anisotropic gradients leads to ellipse-shaped windows.
Fig. 9. Synthetic displacement fields (arrows) and some of the adapted windows (ellipses). Constant vector fields (upper and lower region) enclose a transition zone (middle). Affine transition zone: Identical window shapes for different vector fields: (a) displacement gradients perpendicular to the flow, (b) superimposed by a constant field, and (c) gradients parallel to flow. Sinusoid: (d) Transition zone is sinus-shaped ([0, π]). The adaptation scheme aligns the windows perpendicular to the displacement gradient and reduces the size along the transition direction to avoid smoothing out the boundary.

Fig. 10. Synthetic displacement fields and some of the adapted windows (ellipses), with sharp discontinuities between the inner and outer motion regions. Constant flows: (a) constant vector fields, square boundaries and (b) constant vector fields, round boundaries. Rotational (affine) flows: (c) affine vector fields, square boundaries and (d) affine vector fields, round boundaries. The adaptation scheme reduces the window sizes near the region boundaries to avoid smoothing over motion discontinuities.

be observed near the upper and lower image boundaries, where windows are extremely compressed.

Additive image noise and unpaired particles in combination with the small structures renders the data set Sinusoids II a challenge for any motion measurement algorithm. Again we estimated a coarse description using fixed windows (radius 8 px). Initialised by this result, we run the adaptive approach with the same parameters as for the previous data set, but doubled σ and did not use multiscale calculation.

Figure 12 visualises the resulting displacements and adapted windows. Just as for Sinusoids I, the fixed-window approach can only capture the rough motion structures, but the following adaptive approach complements the details even for the smallest wavelength. However, more outliers than for Sinusoids I can be observed, where also the adapted windows deviate from their expected vertical alignment.

Finally, we compare our results for Sinusoids II to 19 approaches for flow measurement, which were described and benchmarked in [28]. For this purpose we evaluated our method with the same criterion, which is defined as follows: for each sinusoid wavelength λ, we gained a motion profile $u_\lambda(x)$ by averaging the displacements along the vertical axis. Stripes of 10 px at the upper and lower boundary were excluded before. Then using the ground truth profile $u^*_\lambda$, the amplitude ratio was calculated as

$$A(\lambda) := \frac{\int_{-\lambda/4}^{+\lambda/4} u_\lambda(x) \, dx}{\int_{-\lambda/4}^{+\lambda/4} u^*_\lambda(x) \, dx}.$$

The characteristic curve was accurately included into a copy of the comparison plots of the evaluation paper and is presented in Fig. 13. Especially at the lowest wavelength, corresponding to the smallest structures in the data, our adaptive approach outperforms most of the competing implementations.

D. Real Turbulent Experimental Data

Finally, we apply the proposed approach to real PIV data, provided by Johan Carlier in [29] and available at [30]. The experiment describes the turbulent flow behind a cylinder. We chose the image pair number 600 of the data set, recorded with a time difference of 200 µs. Each has a resolution of 1280 px × 1024 px and dynamics of 12 bit. In Fig. 2a a detailed view of the image data attests the low image quality. An overview over the flow is presented in Fig. 14. As in the
previous experiments, we calculated a coarse vector field using fixed windows \((r = 30)\) on the scales \(\{1, 2, 4, 8, 16, 32\}\), see Fig. 15a. Initialised by this result, the adaptive approach delivers a more detailed estimation (Fig. 15b). The windows were constrained to \(r \in [3, 50]\), and \(\sigma\) was chosen as 100. Only the finest scale was used.

Lacking ground truth data, we employ a vector field calculated by the Lavision (http://www.lavision.de/en/) company using their PIV-software Davis as reference (Fig. 15c). Our approach smoothes the displacements in regions of homogeneous motion, while the reference solution exhibits some noise. In turbulent regions, however, the windows are adapted such that gradients in the vector field are prevailed.

In order to separate effects of the window adaptation from the influence of initialisation, we rerun the adaptive method with the same parameter but initialised with the reference solution. The result in Fig. 15d shows the same properties as discussed for the one in Fig. 15b. Furthermore, the coarse structure is almost identical in all three solutions, which suggests their correctness, and shows the robustness of our approach with respect to local optima.

Finally, Fig. 14 marks the location of three regions for which we give detailed views of the vector field together with some of the adapted windows. Figure 3 demonstrates how windows sizes reduce in vicinity of a vortex compared to a homogeneous region. The ability to continuously control the window orientation is beneficial, for example around the vortex in Fig. 16a. As already demonstrated in Fig. 9, the windows do not necessarily align with the direction of the flow, but with its gradient as in Fig. 16b.

V. Conclusion

An adaptive approach to measure motion in PIV image data was presented. It is based on the correlation similarity measure, which has proven to be robust also for noisy data.
Fig. 13. Synthetic vector field, PIV-Challenge 2005, Case A4, region Sinusoids II: Part 1 and 2 of the comparison of the amplitude response depending on the structure wavelength $\lambda$. We included the measurement of our experiments into the plot hard-copied from [28, Fig. 21a] (we redraw the numbers of the horizontal axis for better readability) where 19 flow measurement implementations were compared. Most approaches use cross-correlation while the methods labelled by CLIPS-8 and CEMAGREF-16 are based on optical flow. ESI and *-PTV are particle tracking velocimetry methods. For a description of the competing approaches we refer to [28]. Our approach outperforms most of the other implementations, especially the accuracy for very small structures was improved by using adaptive windows.

Fig. 14. Real 2D PIV experiment: Overview over the fluid flow, determined by our adaptive correlation method. A mean vector field of about 12 px to the right was subtracted everywhere. The rectangles mark the location of detailed views in – from left to right – Fig. 3, 16a and 16b.

In contrast to classical methods which use a discrete search to find the optimal displacement, we formulate it as a variational problem and use continuous optimisation methods. Furthermore, we employ Gaussian-shaped weighting functions whose shape can be continuously controlled and propose a sound adaptation criterion which is based on an error model. Both the displacement measurement and the window adaptation are formulated as two interdependent optimisation problems.

In our experiments we demonstrated the ability of the error model to improve the measurement accuracy and demonstrated the basic behaviour of the adaptation method. We applied our approach to a synthetic PIV benchmark data set and outperformed most of 19 implementations of motion estimators. Finally we showed, that our approach is capable of handling noisy image data from a real experiment. The window adaptation improves the reconstructed vector field in both homogeneous and turbulent regions.

Further work includes to improve the error model function, e.g. to incorporate spatial varying influences (seeding density, image noise level) and further expert knowledge. Regularisation terms could be added to the motion estimation to incorporate prior knowledge, e.g. non-compressibility, on the observed physical process.

APPENDIX

DERIVATION OF THE NOISE TERM

The term $E_{\text{noise}}(\Sigma)$ in (9) describes the impact of image sensor noise and unpaired particles on the accuracy. For this purpose we assume that the measurement in $x$ is a least-squared solution $\hat{u}$ of independent measurements $u(y)$, weighted with the same window function as the one used during correlation. For simplicity, and without loss of generality, we assume that the estimation is centred in $x = 0$. Furthermore, here we assume an unbounded variable domain $\Omega = \mathbb{R}^2$.

$$\hat{u} := \arg \min_{u \in \mathbb{R}^2} \int_{\mathbb{R}^2} w(y, \Sigma) \| u - u(y) \|^2 \, dy$$

$$= \int_{\mathbb{R}^2} w(y, \Sigma) u(y) \, dy = \int_{\mathbb{R}^2} G(\Sigma, y) u(y) \, dy$$

(15)

The noise term should only describe the influence of disturbances in the image data, but not the error caused by inhomogeneous motion. Thus, we assume each measurement to be distributed around the true displacement $u^*$, but disturbed by additive Gaussian noise, i.e. $u(y) \sim \mathcal{N} \left( u^*, \sigma^2 \frac{1}{|A|} \right)$. The constant $\sigma$ is the relative expected error with respect to the size of the domain $A$ on which a single estimation is based on. Then we define the noise term to be the expected square deviation of (15) from the true solution:

$$E_{\text{noise}}(\Sigma) := \mathcal{E} \left\{ \| \hat{u} - u^* \|^2 \right\}$$

(16)

It is possible to derive a closed form expression for this term. To this end we represent the integral in (15) over an infinite
chosen as \( y_{ni} \in A_{ni} \). Then we can define
\[
\hat{u}_n := \sum_{i=1}^{N_n} |A_{ni}| G(\Sigma, y_{ni}) \mathbf{u}(y_{ni}) = \sum_{i=1}^{N_n} w_{ni} \mathbf{u}(y_{ni})
\]
with \( w_{ni} := |A_{ni}| G(\Sigma, y_{ni}) \). In this formulation, the estimated displacement \( \hat{u}_n \) is a linear combination of normally distributed variables and thus is normally distributed as well, i.e. \( \hat{u}_n \sim \mathcal{N}(\mu_n, s_n) \) with:
\[
\mu_n := \mathcal{E}\{\hat{u}_n\} = \sum_{i=1}^{N_n} w_{ni} \mathbf{u}^* = \mathbf{u}^* \sum_{i=1}^{N_n} |A_{ni}| G(\Sigma, y_{ni})
\]
\[
s_n := \mathcal{E}\{(\hat{u}_n - \mu_n)(\hat{u}_n - \mu_n)^\top\} = \sum_{i=1}^{N_n} w_{ni}^2 \frac{\sigma^2}{|A_{ni}|} I
\]
\[
= \sigma^2 I \sum_{i=1}^{N_n} G(\Sigma, y_{ni})^2 |A_{ni}|
\]
Using \( G(\Sigma, x)^2 = \frac{(2\pi \sqrt{\det(2\Sigma)})^{-1} G(\frac{1}{2} \Sigma, x)}{\text{see, e.g., } [31, \text{ eq. (348)}]} \) we obtain
\[
s_n = \frac{\sigma^2}{4\pi \sqrt{\det \Sigma}} I \left( \sum_{i=1}^{N_n} G\left(\frac{1}{2} \Sigma, y_{ni}\right) |A_{ni}| \right).
\]
We assume that for large \( n \) the distribution of \( \hat{u}_n \) describes the distribution \( \hat{u} \sim \mathcal{N}(\hat{\mu}, \hat{\Sigma}) \) well. Passing the limit, we get the expected result for the mean,
\[
\hat{\mu} := \lim_{|A| \to \infty} \lim_{n \to \infty} \mu_n = \mathbf{u}^* \lim_{|A| \to \infty} \int_A G(\Sigma, y) \, dy = \mathbf{u}^*,
\]
and – more importantly – the variance
\[
\hat{\Sigma} := \lim_{|A| \to \infty} \lim_{n \to \infty} s_n = \frac{\sigma^2}{4\pi \sqrt{\det \Sigma}} I \lim_{|A| \to \infty} \int_A G\left(\frac{1}{2} \Sigma, y\right) \, dy
\]
\[
= \frac{\sigma^2}{4\pi \sqrt{\det \Sigma}} I.
\]
Then the definition (16) simplifies to (using [31, eq. (357)])
\[
E_{\text{noise}}(\Sigma) = \mathcal{E}\{\|\hat{u} - \mathbf{u}^*\|_2^2\} = \text{tr} \hat{\Sigma} = \frac{\sigma^2}{2\pi \sqrt{\det \Sigma}}.
\]
The noise level \( \sigma \) is the only parameter for this term.

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REFERENCES


