TomoPIV meets Compressed Sensing

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Abstract. We study the discrete tomography problem in Experimental Fluid Dynamics – Tomographic Particle Image Velocimetry (TomoPIV) – from the viewpoint of Compressed Sensing (CS). The problem results in an ill-posed image reconstruction problem due to undersampling. Ill-posedness is also intimately connected to the particle density. Higher densities ease subsequent flow estimation but also aggravate ill-posedness of the reconstruction problem. A theoretical investigation of this trade-off is studied in the present work.

Keywords: compressed sensing, underdetermined systems of linear equations, positivity constraints in ill-posed problems, sparsest solution, TomoPIV

BACKGROUND AND GOALS

Among the different 3D techniques presently available for measuring velocities of fluids, *Tomographic Particle Image Velocimetry (TomoPIV)* [1] has recently received most attention, due to its increased seeding density with respect to other 3D PIV methods. This, in turn, enables high-resolution velocity field estimates of turbulent flows by means of a cross correlation technique.

TomoPIV is based on a multiple camera-system, three-dimensional volume illumination and subsequent 3D reconstruction, see Figure 1. TomoPIV, in contrast to medical imaging, employs only few projections due to both limited optical access to wind and water tunnels and cost and complexity of the necessary measurement apparatus. As a consequence, the reconstruction problem becomes severely ill-posed, and both the mathematical analysis and the design of algorithms fundamentally differ from the standard scenarios of medical imaging.

A crucial parameter for 3D fluid flow estimation from image measurements is particle density. This parameter also largely influences the tomographical reconstruction problem. Higher densities ease subsequent flow estimation and increase the resolution and measurement accuracy. However, higher densities also aggravate ill-posedness of the reconstruction problem. A thorough investigation of this trade-off is lacking. Our objective is to address this problem taking into account relevant developments in applied mathematics.

METHODS AND RESULTS

The reconstruction of particle volume functions from few projections can be modeled as finding the sparsest solution of an underdetermined linear system of equations, since the original particle distribution can be well approximated with only a very small number of active basis functions relative to the number of possible particle positions in a 3D domain. In general the search for the sparsest solution is intractable (NP-hard), however. The newly developed theory of Compressed Sensing [3, 4, 5] shows that one can compute via ℓ_1-minimization or linear programming the sparsest solution for underdetermined systems of equations provided that the measurement ensemble (the coefficient matrix) satisfies certain conditions. Testing these conditions on generic matrices is often harder than solving the underlying combinatorial ℓ_0-problem as it also implies solving a combinatorial problem which is intractable given the huge dimensionality of the measurement matrix within the TomoPIV setting. However, we showed in [2] that all currently available recovery conditions predict an extremely poor performance of the TomoPIV measurement ensemble when we restrict to a simple but realistic setup geometry.

On average, such matrices perform approximately ten times worse than the Gaussian ensemble which is optimal in the sense that it allows maximal sparsity such that for all less sparse vectors exact recovery is still guaranteed. However, when we slightly perturb the entries of such a degenerate measurement matrix we can boost both worst case and expected reconstruction performance. Then the particle density can be increased by a factor of three while

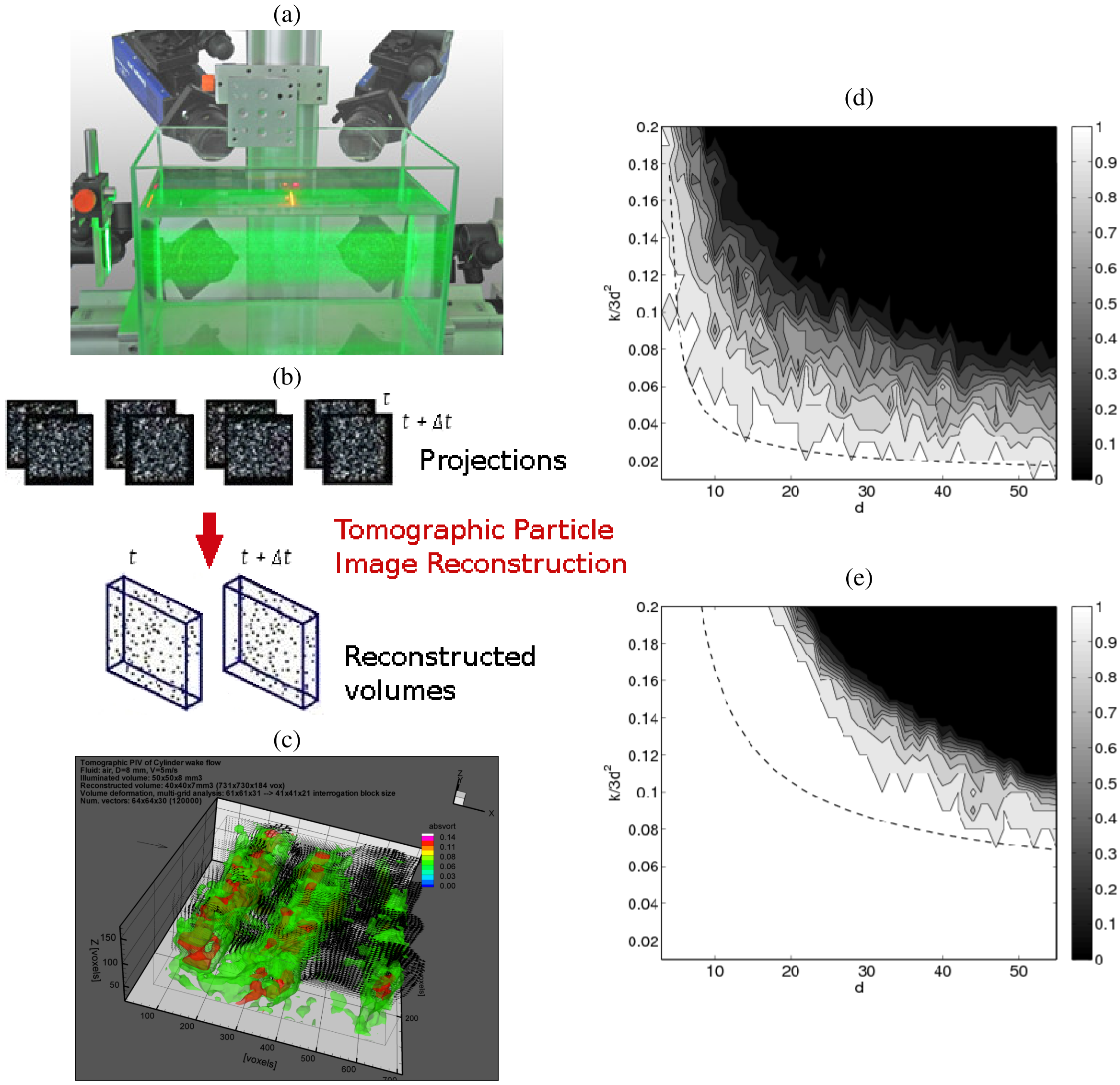

FIGURE 1. Left: The working principle of TomoPIV; (a) Seeding particles within the measurement volume are illuminated by a pulsed light source, and the scattered light pattern is recorded simultaneously from several viewing directions using CCD cameras. (b) The 3D particle distribution (the object) is reconstructed as a 3D light intensity distribution from its projections (2D images) on the CCD arrays. (c) The particle displacement (hence velocity) within the interrogation volume is then obtained by the 3D cross-correlation of the reconstructed particle distribution at the two exposures. Right: Recovery via a standard TomoPIV measurement system (d) versus recovery via the improved measurement system (e). (d) Success and failure empirical phase transition for the standard measurement system along with the phase transition for the optimal measurement system from the viewpoint of CS (the Gaussian ensemble) scaled by factor 0.1 (dashed curve). (e) Success and failure empirical phase transition for the improved measurement system along with the phase transition for the Gaussian measurement ensemble scaled by factor 0.4 (dashed). The results indicate that at least a three times better reconstruction performance is obtained within the considered range of image resolution. For the analytical phase transitions for arbitrary high resolution (parameter d) see [2].

preserving the number of measurements.

In a nutshell, we show that the TomoPIV problem is quite degenerate from the viewpoint of compressed sensing, thus leading to poor performance guarantees. On the other hand, the probabilistic analysis of [2] yields average performance bounds that back up current rules of thumb of engineers for choosing particle densities in practice.

OUTLOOK & FUTURE WORK

We currently study the tomographic problem of reconstructing particle volume functions from the general viewpoint of compressed sensing. We show that all currently available CS concepts predict an extremely poor worst case performance, and a low expected performance of the TomoPIV measurement system, indicating why low particle densities only are currently used by engineers in practice. Simulations demonstrate however that slight random perturbations of the TomoPIV measurement matrix considerably boost both worst-case and expected reconstruction performance. This finding is interesting for CS theory and for the design of TomoPIV measurement systems in practice. Our work aims at pointing out connections between the fields of compressed sensing and discrete tomography in order to stimulate further research.

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