Exercises – Sheet 1

Exercise 1

Consider $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two normed spaces. Show that for a linear mapping $F: X \to Y$ the following statements are equivalent

(a) F is continuous on X;

(b) F is continuous in $0 \in X$;

(c) F is bounded, i.e. there exists a constant $C > 0$ with

$$\|Fx\|_Y \leq C\|x\|_X\,, \quad \forall x \in X\,.$$

Exercise 2

Consider

$$C^0([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\},$$

the space of continuous real-valued functions defined on the unit interval. Set $X = Y = C^0([0,1])$, but equip X with the L^1 norm and Y with the supremum norm. Show that the embedding of X into Y is discontinuous.

Hint: Consider the sequence

$$f_n(x) = \begin{cases} -n^2x + n, & 0 \leq x \leq 1/n, \\ 0, & \text{otherwise,} \end{cases}$$

and compute $\|f_n\|_{L^1}$ and $\sup_{x\in[0,1]} |f_n(x)|$.

Exercise 3

Consider the map $F : \ell^2 \to \ell^2$ defined by

$$Fx = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right).$$

Show that F is bounded, but that the range of F is not closed.

Exercise 4

Use the corollary 1.3 of the Hahn-Banach theorem to show that if $f(x) = f(y)$ for all $f \in X^*$ we necessarily have $x = y$.

Exercise 5

Show the following basic properties of weak convergence

(a) If $x^k \to x$ than $x^k \rightharpoonup x$.

(b) The weak limits are unique.

Hint: To show (b) you can use the previous exercise 4.

Exercise 6

Can you specify a closed unit ball that is not compact?

Exercise 7

This is the first practical exercise that you are required to solve.[1]

(a) Load image "input1.jpg" in Python (`imread` from matplotlib.pyplot) from Teams or the tutorial webpage `https://ipa.math.uni-heidelberg.de/dokuwiki/doku.php?id=teaching:st21:ueb:mb`. What are the spatial dimensions, data type and range of the image? How many color channels has the image?

(b) Scale the colour values in the range $[0, 1]^3$ and display the new scaled image (`imshow` from matplotlib.pyplot).

(c) Calculate the mean colour value (should be a vector with three values) over all pixels.

(d) What is the colour of the pixel at position $(299, 11)$?
Note: the top left pixel has the coordinate $(0, 0)$.

(e) Convert the image in grayscale and display it. The gray value h can be calculated from the RGB values using $h = 0.3r + 0.59g + 0.11b$. What are the minimum and maximum gray values in the image?

(f) Save the grayscale image in the PNG-format (`imsave` from matplotlib.pyplot).

(g) Create a histogram of the grayscale image (`hist` from matplotlib.pyplot).

[1] Please upload your solution to Teams or send it to Matthias Zisler `zisler@math.uni-heidelberg.de` before the next tutorial on the 28th of April. You can use either Python or Matlab.